

Vortex density waves and high-frequency second sound in superfluid turbulence hydrodynamics

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Abstract

In this paper we show that a recent hydrodynamical model of superfluid turbulence describes vortex density waves and their effects on the speed of high-frequency second sound. In this frequency regime, the vortex dynamics is not purely diffusive, as for low frequencies, but exhibits ondulatory features, whose influence on the second sound is here explored.

Hydrodynamics of superfluid turbulence, characterized by a tangle of quantized vortex lines [1, 2], aims to describe the couplings between pressure and heat perturbations, and the vortex density dynamics [3]-[7]. Such hydrodynamics is a lively topic, with recent emphasis, for instance, on nonlinear features such as the influence of intense heat pulses on the vortex tangle [8]-[10], or in multi-scale formulations allowing to eliminate the fast processes to derive evolution equations for the slow processes [11]. In the present paper we will focus our attention on a different topic, namely, the behaviour of linear heat waves and vortex density waves in the high-frequency domain. This has not received much previous attention, but we think it is worthwhile because, whereas at low frequencies the behaviour of the vortex distortions is mainly diffusive, at high enough frequencies, the vortex lines behave as an elastic medium, and are able to propagate waves by themselves, i.e. they behave like a viscoelastic medium: diffusive at low frequencies, elastic at high frequencies. Their effect on second sound propagation is of much interest to provide a suitable physical interpretation of the experimental data based on second sound.

Collective vortex waves in rotating superfluids have been studied in depth for a long time [12], and the so-called Tkachenko transverse elastic waves in the vortex arrays with crystalline order arising in rotating cylinders have been theoretically discussed since 1966. However, vortex density waves in counterflow situations have not been studied, up to our knowledge.

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Usually, one considers homogeneous vortex tangles in counterflow situation (i.e. under vanishing barycentric speed) and the evolution of L , the *vortex-line density*, is assumed to be the well-known Vinen's equation [1, 2, 6, 13]

$$\frac{dL}{dt} = \sigma_L = AqL^{3/2} - BL^2, \quad (1)$$

where q is the absolute value of the heat flux \mathbf{q} , A and B parameters linked to the dimensionless coefficients appearing in Vinen's equation by the relations $A = \alpha_v/\rho_s Ts$ and $B = \kappa\beta_v$ [7], ρ_s being the density mass of the superfluid component, T the absolute temperature, s the entropy, $\kappa = h/m$ the quantum of vorticity (m the mass of the ^4He atom and h Planck's constant, $\kappa \simeq 9.97 \cdot 10^{-4} \text{cm}^2/\text{s}$).

This equation is used to describe homogeneous turbulence. When the vortex line density is inhomogeneous, such an equation must be generalized incorporating nonlocal terms. In [7], a set of evolution equation for ϵ , \mathbf{q} and L , with ϵ being the energy density, was proposed with special attention to their consistency with the second law of thermodynamics through the Liu's procedure. Such equations were [7]

$$\begin{cases} \rho\dot{\epsilon} + \nabla \cdot \mathbf{q} = 0, \\ \dot{\mathbf{q}} + \zeta_0 \nabla T + \chi_0 \nabla L = -KL\mathbf{q}, \\ \dot{L} + \nabla \cdot (\nu_0 \mathbf{q}) = \sigma_L. \end{cases} \quad (2)$$

The coefficient K is related to the Hall-Vinen friction coefficient B_{HV} as $K = \frac{1}{3}\kappa B_{HV}$, ν_0 and χ_0 are suitable coefficients whose physical meaning we want to explore and ζ_0 is the ratio of the heat conductivity of the fluid over the relaxation time of the heat flux, which is given by $(KL)^{-1}$. A further term proportional to $L^{3/2}\hat{\mathbf{q}}$ may be added to the right-hand side of the equation (2b) for the heat flux, describing a dry friction term. Here, we will omit it for the sake of simplicity [6, 7]. For vanishing χ_0 and ν_0 , the two first equations describe second sound (temperature waves) with speed $V_2^2 = \frac{\zeta_0}{\rho c_V}$, with c_V the specific heat per unit mass at constant volume, and the third equation reduces to Vinen's equation (1). In the steady state, the second equation reduces to

$$\mathbf{q} = -\frac{\zeta_0}{KL} \nabla T - \frac{\chi_0}{KL} \nabla L. \quad (3)$$

The coefficient of the first term is just the heat conductivity already mentioned, and the second term describes a coupling between the vortex line density and the heat flux, analogous in some way to the coupling between concentration gradient and heat flux in usual fluids (the so-called Soret effect). The term in ∇L could be related to the internal energy density of the tangle, which is given by $\epsilon_V L$, with $\epsilon_V = (\rho_s \kappa^2 / 4\pi) \ln(c/a_0 L^{1/2})$ the energy per unit length of the vortices. In more specific terms, it may be related to the tangle contribution to the total pressure, as given in (10). Since in this paper we are concerned with high frequency and short wavelength perturbations, the contribution of ∇L will be relevant, in contrast with usual situations at long wavelengths.

From (2) it is seen that when $\dot{\mathbf{q}}$ is neglected and the system is isothermal, and in the linear approximation, one obtains from (2b) and (2c) a reaction-diffusion equation for L of the form

$$\dot{L} = \frac{\nu_0 \chi_0}{KL_0} \nabla^2 L + \sigma_L, \quad (4)$$

whose diffusion coefficient D is given by $D = \frac{\nu_0 \chi_0}{KL_0}$.

If, instead, $\dot{\mathbf{q}}$ is not neglected in (2b) and still in the isothermal assumption – for the sake of simplicity – we obtain

$$\ddot{L} + KL_0\dot{L} = \nu_0\chi_0\nabla^2 L + KL_0\sigma_L + \dot{\sigma}_L. \quad (5)$$

At high frequencies, i.e. for $\omega \gg KL_0$ and much higher than the inverse of the characteristic time of the vortex destruction and formation as described by σ_L , namely $\omega \gg 2BL_0 - \frac{3}{2}AqL_0^{1/2}$, (5) becomes

$$\ddot{L} \approx \nu_0\chi_0\nabla^2 L, \quad (6)$$

that is, we get vortex waves with speed

$$v_\infty^2 = \nu_0\chi_0. \quad (7)$$

This outlines the physical features of the coefficients $\nu_0\chi_0$ as their product is seen to be related both to the diffusion coefficient and to the speed of vortex waves.

Elastic compressional waves in the vortex density obtained in (6) may also be interpreted in a more mechanistic way. First of all, we may interpret the term $\nu_0\mathbf{q}$ in equation (2c) as $\nu_0\mathbf{q} = L\mathbf{v}_L$, with \mathbf{v}_L an effective velocity of the tangle because the divergence term in the balance equation (2c) may be interpreted as a flux of vortex lines moving with peculiar velocity \mathbf{v}_L . This effective velocity will have an evolution equation of the type

$$\rho_{eff} \frac{\partial \mathbf{v}_L}{\partial t} = -\nabla p_V, \quad (8)$$

where ρ_{eff} is an effective density of the vortices and p_V is the vortex contribution to the pressure. In fact, vortices themselves do not have mass, but they have indirectly associated an inertia due to the mass of the surrounding rotating fluid. By following Sonin (page 94 of Ref. 12) we estimate this mass per unit volume as $\rho_s L r^2$, with ρ_s the density of the superfluid component, which is the one which participates in the rotation around the vortices of the fluid and r a characteristic radius of the zone in which the superfluid is affected by the motion of the vortex; this will be of the order of the average vortex separation, namely $L^{-1/2}$; in this way, we have $\rho_{eff} \propto \rho_s$. This is the reason that ρ_s is the density which appears in the coefficient ϵ_V appearing in the expression for the vortex density of the tangle, as given in the paragraph below equation (3).

Then, we may combine equation (8) with equation (2c), which at high enough frequency, when the influence of the production-destruction term σ_L is negligible, yields

$$\frac{\partial^2 L}{\partial t^2} = -L_0 \nabla \cdot \frac{\partial \mathbf{v}_L}{\partial t} = \frac{L_0}{\rho_{eff}} \nabla^2 p_V. \quad (9)$$

The total pressure of the turbulent superfluid has the form (equation (4.16) of Ref. 7)

$$p = p^* + \epsilon_V L, \quad (10)$$

p^* being the pressure of the bulk superfluid and $\epsilon_V L$ the contribution of the tangle, with ϵ_V the energy per unit length of the vortices. Combination of (9) and (10) and taking into account that $\rho_{eff} \propto \rho_s$, yields

$$\frac{\partial^2 L}{\partial t^2} = \left(L_0 \frac{\epsilon_V}{\rho_{eff}} \right) \nabla^2 L. \quad (11)$$

Then the velocity of the vortex waves will be

$$v_L^2 = \left(L_0 \frac{\epsilon_V}{\rho_{eff}} \right) \propto \frac{L_0 \kappa^2}{4\pi} \ln \left(\frac{c}{a_0 L^{1/2}} \right). \quad (12)$$

The combination $L_0 \kappa^2$ is similar to the combination $\Omega \kappa$ appearing in the velocity of inertial waves in the vortex array in rotating superfluids [12], if one replaces $L_0 = 2\Omega/\kappa$, Ω being the angular speed of the rotating cylinder.

We now go to the diffusion coefficient D corresponding to equation (4). Since we have obtained an estimation of $\nu_0 \chi_0$ appearing in (6), we will take advantage of it to obtain an expression for D . We get

$$D = \frac{\nu_0 \chi_0}{K L_0} \propto \frac{\kappa}{B_{HV} 4\pi} \ln \left(\frac{c}{a_0 L^{1/2}} \right) \quad (13)$$

Indeed, it was known that $D \propto \kappa$ on dimensional grounds and on some numerical simulation [14]. Here, we have a more explicit expression, based on a more microscopic model. Unfortunately, our derivation cannot set precisely the proportionality constant in (13) due to the ambiguity in the mass density ρ_{eff} associated to the vortices (more precisely, to the superfluid associated to the vortices). The analysis of the high-frequency regime may be indeed rewarding in new results which are not available if the study is limited to the low-frequency regime.

Up to here we have considered an isothermal situation. According to (2), the behavior of \mathbf{q} and L under non-isothermal conditions is connected to the behavior of the field T in such a way that the complete study of the system (2) is required. In [7] two of us have already tempted to solve this more complicated situation, but now to the light of which we have said before, some more explicit conclusions may be done with a clearer physical meaning in the high frequency regime, which complements the information obtained at low frequencies.

As in [7], expressing the energy in terms of the vortex line L and the temperature T and linearizing the two contributions σ_L and $L\mathbf{q}$ around the stationary solutions, already found in [7],

$$\mathbf{q} = \mathbf{q}_0 = (q_{10}, 0, 0), \quad L = L_0 = \frac{A^2}{B^2} [q_{10}]^2, \quad T = T_0(\mathbf{x}) = T^* - \frac{KLq_{10}}{\zeta_0} x_1, \quad (14)$$

the system (2) assumes the following form

$$\begin{cases} \rho c_V \dot{T} + \rho \epsilon_L \dot{L} + \nabla \cdot \mathbf{q} = 0, \\ \dot{\mathbf{q}} + \zeta_0 \nabla T + \chi_0 \nabla L = -K[L_0 \mathbf{q} + \mathbf{q}_0(L - L_0)], \\ \dot{L} + \nu_0 \nabla \cdot \mathbf{q} = - \left[2BL_0 - \frac{3}{2} A q_{10} L_0^{1/2} \right] (L - L_0) + A q_{10} L_0^{3/2} (q_1 - q_{10}). \end{cases} \quad (15)$$

Now, if we suppose the propagation of harmonic plane waves of the form

$$\begin{cases} T = T_0(\mathbf{x}) + \tilde{T} e^{i(\bar{K}\mathbf{n}\cdot\mathbf{x} - \omega t)} \\ \mathbf{q} = \mathbf{q}_0 + \tilde{\mathbf{q}} e^{i(\bar{K}\mathbf{n}\cdot\mathbf{x} - \omega t)} \\ L = L_0 + \tilde{L} e^{i(\bar{K}\mathbf{n}\cdot\mathbf{x} - \omega t)}, \end{cases} \quad (16)$$

where $\bar{K} = k_r + ik_s$ is the complex wave number, ω is the frequency and \mathbf{n} the unit vector in the direction of the wave propagation, then we have the propagation of waves (vortex waves and heat waves) which move at the same speed $w_2 = \frac{\omega}{k_r}$ given by the combination of the second

sound and vortex waves velocities. As in [7], we are assuming that the quantities oversigned by a tilde are small ones, and whose product can be neglected.

Inserting (16) in the linearized system (15) we obtain the following algebraic system for the quantities \tilde{T} , \tilde{L} and $\tilde{\mathbf{q}}$

$$\begin{cases} -[\rho c_V]_0 \omega \tilde{T} - [\rho \epsilon_L]_0 \omega \tilde{L} + \bar{K} \tilde{\mathbf{q}} \cdot \mathbf{n} = 0, \\ (-\omega - iN_1) \tilde{\mathbf{q}} + \bar{K} [\zeta_0]_0 \tilde{T} \mathbf{n} + (\bar{K} [\chi_0]_0 \mathbf{n} - iN_3 \hat{\mathbf{q}}_0) \tilde{L} = 0, \\ (-\omega - iN_2) \tilde{L} + \bar{K} [\nu_0]_0 \tilde{\mathbf{q}} \cdot \mathbf{n} + iN_4 \tilde{q}_1 = 0, \end{cases} \quad (17)$$

where

$$N_1 = KL_0, \quad N_2 = 2BL_0 - \frac{3}{2}AL_0^{1/2}q_{10}, \quad N_3 = Kq_{10}, \quad N_4 = Aq_{10}L_0^{3/2},$$

and the subscript 0, which will be deleted from now on, denotes quantities referring to the unperturbed states.

Now, let consider that the direction of the wave propagation is collinear to the initial heat flux, i.e. $\mathbf{n} = (1, 0, 0)$, then the system (17) becomes

$$\begin{cases} -\rho c_V \omega \tilde{T} - \rho \epsilon_L \omega \tilde{L} + \bar{K} \tilde{q}_1 = 0, \\ (-\omega - iN_1) \tilde{q}_1 + \bar{K} \zeta_0 \tilde{T} + (\bar{K} \chi_0 - iN_3) \tilde{L} = 0, \\ (-\omega - iN_1) \tilde{q}_2 = 0, \\ (-\omega - iN_1) \tilde{q}_3 = 0, \\ (-\omega - iN_2) \tilde{L} + (\bar{K} \nu_0 + iN_4) \tilde{q}_1 = 0. \end{cases} \quad (18)$$

In the hypothesis of high-frequency waves, which means $\omega \gg N_1$, $\omega \gg N_2$ and $|\bar{K}| \gg \max\left(\frac{Kq_{10}}{\chi_0}, \frac{Aq_{10}L_0^{3/2}}{\nu_0}\right)$, the previous algebraic system (18) becomes

$$\begin{cases} -\rho c_V \omega \tilde{T} - \rho \epsilon_L \omega \tilde{L} + k_r \tilde{q}_1 = 0, \\ -\omega \tilde{q}_1 + k_r \zeta_0 \tilde{T} + k_r \chi_0 \tilde{L} = 0, \\ -\omega \tilde{L} + k_r \nu_0 \tilde{q}_1 = 0, \\ -\omega \tilde{q}_2 = 0, \\ -\omega \tilde{q}_3 = 0. \end{cases} \quad (19)$$

The above system has nontrivial solutions if and only its determinant is zero, which corresponds to the following dispersion relation

$$w_2^2 = V_2^2(1 - \nu_0 \rho \epsilon_L) + v_\infty^2, \quad (20)$$

where $w_2 = \omega/k_r$ is the speed of the wave, $V_2^2 = \zeta_0/\rho c_V$ is the second sound speed in the absence of vortices and $v_\infty^2 = \chi_0 \nu_0$ is the speed of the vortex wave, which we have found in (7). We recall that all three fields T , L and \mathbf{q} vibrate with the same speed w_2 given by (20) in such a way that each field contributes to the vibrations of the other two. If we try to read the relation (20) in terms of the second sound, we note that the vortex vibrations modify this second sound speed through the two contributions $-V_2^2 \nu_0 \rho \epsilon_L$ and v_∞^2 , the latter due to the presence of the vortex waves and the former due to the reciprocal existence of two waves. The same conclusion may be achieved reading the relation (20) in terms of the vortex waves. The correction for the speed of the second sound is not important, because V_2 is of the order of 20 m/s near 1,7 K [9, 11], whereas, for $L_0 = 10^6 \text{cm}^{-2}$, and according to the estimation (12), the speed of vortex density waves would be of the order 0,25 cm/s, much lower than V_2 .

In the earlier analysis of the system (15), we have only considered the terms of the equations in which ω and k_r appear such as we have also neglected the term k_s relative to the dissipation

of the wave. Now, we assume that the quantities N_1 , N_2 , N_3 and N_4 are coefficients small enough to assume them as perturbations of the physical system. This is reasonable at high-frequencies, since we have assumed that $\omega \gg N_1$, $\omega \gg N_2$ and $|\bar{K}| \gg \max\left(\frac{Kq_{10}}{\chi_0}, \frac{Aq_{10}L_0^{3/2}}{\nu_0}\right)$. Thus, the contributions of these coefficients can modify the speed of the wave in a small quantity δ and the imaginary part of the wave number, k_s .

Therefore, let assume that the speed of the wave has the following expression

$$w = \frac{\omega}{k_r} = w_2 + \delta, \quad (21)$$

for which substituting it in the dispersion relation, i.e. the equation obtained imposing that the determinant relative to the system (15) vanishes, we obtain at the lower order the relation (20) for the speed w_2 . From the next order follows that $\delta = 0$, that is the perturbations due to the coefficients N_1 , N_2 , N_3 and N_4 do not modify the speed of the wave while they modify the coefficients k_s related to the attenuation in the form

$$k_s^{\parallel} = \frac{N_2(w_2^2 - V_2^2) + w_2(\rho\epsilon_L N_4 V_2^2 + N_1 w_2 + N_3 \nu_0 - N_4 \chi_0)}{2w_2^3}. \quad (22)$$

Anyway, this modification will be small, because $w_2^2 - V_2^2$ is small and the coefficients N_i are also small in the situation considered.

Now, we consider the case in which \mathbf{n} and the initial heat flux \mathbf{q}_0 are orthogonal, and in particular we choose $\mathbf{n} = (0, 0, 1)$ [7]. Through this choice the system (15) becomes

$$\begin{cases} -\rho c_V \omega \tilde{T} - \rho \epsilon_L \omega \tilde{L} + \bar{K} \tilde{q}_3 = 0, \\ (-\omega - iN_1) \tilde{q}_3 + \bar{K} \zeta_0 \tilde{T} + \bar{K} \chi_0 \tilde{L} = 0, \\ (-\omega - iN_2) \tilde{L} + \bar{K} \nu_0 \tilde{q}_3 + iN_4 \tilde{q}_1 = 0, \\ (-\omega - iN_1) \tilde{q}_1 - iN_3 \tilde{L} = 0, \\ (-\omega - iN_1) \tilde{q}_2 = 0. \end{cases} \quad (23)$$

If now we make the same assumption as the previous case, i.e. high frequencies ω and high wave number k_r , the last system becomes

$$\begin{cases} -\rho c_V \omega \tilde{T} - \rho \epsilon_L \omega \tilde{L} + k_r \tilde{q}_3 = 0, \\ -\omega \tilde{q}_3 + k_r \zeta_0 \tilde{T} + k_r \chi_0 \tilde{L} = 0, \\ -\omega \tilde{L} + k_r \nu_0 \tilde{q}_3 = 0, \\ -\omega \tilde{q}_1 = 0, \\ -\omega \tilde{q}_2 = 0. \end{cases} \quad (24)$$

The dispersion relation of this system is found by setting to zero the discriminant of the matrix associated to (24), from which the same speed $w_2 = \omega/k_r$ of the wave of the previous case is obtained

$$w_2^2 = V_2^2(1 - \nu_0 \rho \epsilon_L) + v_\infty^2. \quad (25)$$

Now, following the same procedure of the previous case, we consider the quantities N_1 , N_2 , N_3 and N_4 as small perturbations of the physical system in such a way the speed of the wave is modified by a quantity δ and the imaginary part cannot be negligible. Therefore, assuming that the new speed of the wave has the form $w = \omega/k_r = w_2 + \delta$, it is found the relation (25)

for the speed w_2 , at the lowest order, and $\delta = 0$ and

$$k_s^\perp = \frac{N_2 (w_2^2 - V_2^2) + N_1 w_2^2}{2w_2^3}, \quad (26)$$

at the successive order. Note that also in this case the perturbations do not modify the speed of the waves but only the dissipative term k_s . From a comparison between the two relations of k_s , (22) and (26), one may note that the quantities N_3 and N_4 take part only in (22), that is, only when the direction of the wave propagation is collinear to the initial heat flux. In particular, one may write $k_s^\parallel = k_s^\perp + \frac{(\rho \epsilon_L V_2^2 - \chi_0) N_4 + N_3 \chi_0}{2w_2^3}$.

In summary, we have shown that at high enough frequencies the dynamics of inhomogeneous vortex tangles shows a crossover from a diffusive to propagation behavior. We have studied the consequences of these contributions on the speed of high-frequency second sound in (20) and (25) and on the attenuation of longitudinal and transverse second sound in (22) and (26). The origin of the longitudinal density waves predicted by the equation (2) is to be found in the vortex contribution to the total thermodynamics pressure of the system, as described in (10). This is different from the transverse elastic waves known as Tkachenko waves found in rotating vortex arrays. An evaluation of the speeds shows that the speed of the second sound, according to (20) and (25), will not be much influenced by the presence of the vortex waves; in contrast, the speed of vortex waves, which would be rather small in the absence of second sound, is much increased in the presence of high-frequency second sound, because in these circumstances they propagate at a common speed (20) or (25). The results for the attenuation coefficient according to (22) and (26) are also interesting, because it turns out that the attenuation of second sound at high frequency will be very low. This is in contrast with what happens at low frequency, or when the vortex tangle is assumed as perfectly rigid, not affected by the second sound, in which case the relative motion of the normal fluid with respect to the vortex lines yields an attenuation which allows to determine the vortex line density L of the tangle [1, 2]. However, the wave character of vortex density perturbations at high frequency makes that vortex lines and the second sound become two simultaneous waves with a low joint dissipation, in the first-order approach. Thus, from the practical point of view, it seems that, at high frequency, second sound will not provide much information on the vortex tangle because the influence of the average vortex line density L_0 is small both in the speed as in the attenuation. We have also obtained an explicit expression for the vortex diffusion coefficient (13) in terms of κ and B_{HV} , which is somewhat analogous to the Einstein expression for the diffusion coefficient, but with the quantum κ of the turbulence instead of the thermal energy and the friction coefficient B_{HV} in the denominator. This may be useful for studies on hydrodynamics of vortex tangles [3, 4, 7].

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